ANSWERS

I. 1. Answers will vary.

2. a) $f(n)=2f(n-1)+1$, with $f(1)=1$

b) $x_5=9.6875$; $x_{10} \approx 9.99$. As n increases, x_n approaches 10. [Enter 5 on your graphing calculator. Then iterate 0.5ANS+5 the desired number of times.]

c) 39. [$x_1=2(3)-1=5$ / $x_2=2(5)-2=8$ / $x_3=2(8)-3=13$ / $x_4=2(13)-4=22$, $x_5=2(22)-5=39$.]

3. a) • Using the recurrence relation: You must iterate five times starting with F_3 and stopping at F_7 ($F_1=F_2=1$ are given.).

• Using the explicit formula: You evaluate the formula for $n=7$.

b) $F_3=2$, $F_4=3$, $F_5=5$.

4. Let T_n be the n^{th} triangular number.

a)

• Recursive definition: $T_n = T_{n-1} + n$, where $T_n=1$.

• Explicit definition: $T_n=n(n+1)/2$

b)

• $T_6=6(7)/2=21$.

• $T_1=1$; $T_2=1+2=3$; $T_3=3+3=6$; $T_4=6+4=10$; $T_5=10+5=15$; and finally, $T_6=15+6=21$.

II. 1. a) In the first, 1 is subtracted from the quantity 2^n; in the second, 1 is subtracted from the exponent of 2, which is n.

b)

n	2^n-1	2^{n-1}
-2	-.75	.125
-1	-.5	.25
0	0	.5
1	1	1
2	3	2
3	7	4

For $x<1$, $2^n-1<2^{n-1}$

For $x>1$, $2^n-1>2^{n-1}$

For $x=1$, $2^n-1=2^{n-1}$

c) Yes.

2. *1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.*

a)

$F_2/F_1=1$; $F_3/F_2=2$; $F_4/F_3=1.5$; $F_5/F_4=1.66..$;
$F_6/F_5=1.6$ $F_7/F_6=1.625$; $F_8/F_7=1.615$;
$F_9/F_8=1.619$; $F_{10}/F_9=1.617$; $F_{11}/F_{10}=1.6182$;
$F_{12}/F_{11}=1.6179$.

Observations:

(i) The ratios "converge" to the golden ratio, $\varphi=1.618034...$, which we saw in Lesson 16.

(ii) The values alternate between being a little greater than and a little less than the exact value of φ.

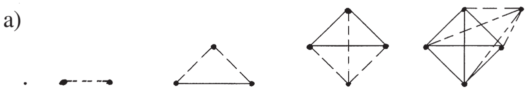
b) The greater the value of n, the closer the decimal value of F_n/F_{n-1} is to φ.

3. a) Because x_n is defined in terms of the three preceding terms x_{n-1}, x_{n-2}, x_{n-3} for $n\geq 3$.

b) $x_3=x_2-x_1+x_0=2-1+0=1$
$x_4=x_3-x_2+x_1=1-2+1=0$
$x_5=x_4-x_3+x_2=0-1+2=1$
$x_6=x_5-x_4+x_3=1-0+1=2$
$x_7=x_6-x_5+x_4=2-1+0=1$

0, 1, 2, 1, 0, 1, 2, 1, 0,...

4. a)



$l_1=0$	$l_2=l_1+1$	$l_3=l_2+2$	$l_4=l_3+3$	$l_5=l_4+4$
	=0+1	-1+2	-3+3	=6+4
	=1	-3	-6	=10

Note: The dotted line segments at stage n are the new line segments added to l_{n-1} to obtain l_n.

b) $l_n=l_{n-1}+(n-1)$ for $n\geq 2$, where $l_1=0$

c) $l_n=n(n-1)/2$

III. 1. a) 1 disk: 1 move
2 disks: 3 moves
3 disks: 7 moves
4 disks: 15 moves

b) $M_n=M_{n-1}+1+M_{n-1}=2M_{n-1}+1$ for $n\geq 2$, where $M_1=1$.

c) $M_n=2^n-1$

ALGEBRA 1

The Complete Course

Lesson Twenty Nine

Section VII: Iterating Functions

Looking at Functions Recursively

KA8459

Worksheet

HOW TO USE THE VIDEO AND WORKSHEET

1. The "STOP TO THINK" signal means pause to think.
2. The "STOP TO WORK" signal means work the problem(s).
3. Rewind the tape and watch the lesson again if the concept is not clear.
4. Students should complete the exercises on the worksheet to confirm their understanding of this lesson.

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In this lesson we mentioned a variety of instances of "Every-day Recursion". Can you find others?

2. In this lesson, we iterated the recurrence relation $x_n = 2x_{n-1}+1$, starting with $x_1 = 1$.

   a) Write this same recurrence relation using function notation.

   b) Suppose we wished to iterate $x_n = (l/2)\ x_{n-1}+5$ starting with $x_1 = 5$. By hand, or with the help of your graphing calculator, find x_5 and x_{10}. What do you notice?

   c) Often, the initial condition (or initial/starting value) is x_0 instead of x_1. Compute x_5 for the recurrence relation $x_n = 2x_{n-1} - n$, where $x_0 = 3$.

3.
   a) Explain the difference between calculating the seventh Fibonacci number using the recurrence relation and the explicit formula.

   b) Practice using the explicit formula by calculating F_3, F_4, F_5. (Use your calculator.)

4. We revisited the triangular numbers and compared their recursive and explicit definitions.

   a) Recall the two ways of generating these numbers.

   b) Compute the 6^{th} triangular number using the explicit formula first, then the recursive one (i.e. the recurrence relation).

II. SUPPLEMENTARY EXERCISES

1. In this lesson, we encountered the sequence $x_n = 2^n - 1$. In a previous lesson, we encountered the expression 2^{n-1}. Let $y_n = 2^{n-1}$. The purpose of this exercise is to appreciate the difference between the two expressions, $x_n = 2^n - 1$ and $y_n = 2^{n-1}$.

   a) Explain, in your own words, the algebraic difference between the quantities $2^n - 1$ and 2^{n-1}.

   b) For n -2,-1, 0, 1, 2, 3, evaluate x_n -2^n-1 and y_n -2^{n-1}. How do they compare?

   c) Graph the functions $f(x) = 2^x - 1$ and $g(x)$-2^{x-1} using your graphing calculator. Do the graphs confirm your answers to part b)?

2. Using the method of your choice, evaluate the first *12* Fibonacci numbers, F_1 through F_{12}.

   a) Compute the successive ratios F_2/F_1, F_3/F_2, F_4/F_3,..., F_{12}/F_{11}, using your calculator. What observations can you make?

   b) What can you conjecture about the value of F_n/F_{n-1}, as n becomes very large?

3. We recursively define $x_n = x_{n-1} - x_{n-2} + x_{n-3}$ for $n \geq 3$ where $x_0 = 0$, $x_1 = 1$, $x_2 = 2$.

   a) Why is this recurrence relation called a third-order recurrence relation?

   b) Calculate the first few terms of the sequence until the pattern is clear. What set of values make up this sequence?

4. Consider from a recursive point of view, the total number of line segments, l_n, needed to connect n given points that are not collinear (i.e., that are not in a straight line).

   a) Calculate l_1 through l_5 indicating the new line segments needed when going from l_{n-1} to l_n. (Make a picture.)

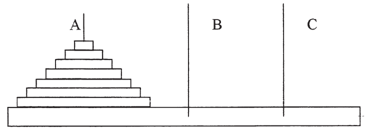
   b) Generalize by writing a recurrence relation for l_n.

   c) Can you find an explicit formula for l_n?

III. INVESTIGATIVE PROBLEM

1. The Tower of Hanoi. The object of this ancient game is to move the tower of seven disks (of graduated size) from peg A to peg C, in the fewest possible moves, following these rules:

   (1) You may only move one disk at a time; and
   (2) You may not place a disk on top of one that is smaller.



   a) What is the minimum number of moves required for *1* disk? *2* disks? *3* disks? *4* disks?

   b) Write a recurrence relation for the minimum number of moves, M_n, required to move n disks.

   c) Can you find an explicit formula for M_n?

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